Explicit formulae for moments of decision times of single- and double-threshold drift-diffusion processes

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Drift Diffusion Model and Free Response Paradigm

- Models human decision making in two alternative choice tasks
- Evidence evolution in a two alternative choice task is modeled by
  \[ dx(t) = a \, dt + \sigma \, dW(t), \quad x(t) = x_0 \]
- Decision process at time \( \tau \) is
  \[
  \begin{cases}
  x(\tau) > z, & \text{choose alternative 1,} \\
  x(\tau) < -z, & \text{choose alternative 2,} \\
  \text{else,} & \text{collect more evidence.}
  \end{cases}
  \]
Single Threshold DDM

1. For a single (upper) threshold $x = z \geq 0$, the DTs follow Wald distribution

$$p(t) = \sqrt{\frac{\eta}{2\pi t^3}} \exp\left(\frac{-z(t - z/a)^2/\sigma^2}{2\alpha^2 t}\right)$$

2. Moments of decision time are

$$E[DT] = \alpha = \frac{z}{a}, \quad \text{Var}[DT] = \frac{z\sigma^2}{a^3}$$

$$CV = \frac{\sqrt{\text{Var}[DT]}}{E[DT]} = \sqrt{\frac{\sigma^2}{az}}, \quad \text{Skew} = 3\sqrt{\frac{\sigma^2}{az}} \quad (= 3 CV)$$
Double Threshold DDM

1. For DDM with symmetric thresholds \( \pm z \), error rate is

\[
ER = \frac{e^{-2k_x} - e^{-2k_z}}{e^{2k_z} - e^{-2k_z}}
\]

where \( k_z = az/\sigma^2 \) and \( k_x = ax_0/\sigma^2 \)

2. expected DT

\[
\mathbb{E}[\text{DT}] = \frac{\sigma^2}{a^2} \left[ k_z \coth(2k_z) - k_z e^{-2k_x} \csch(2k_z) - k_x \right].
\]
Double Threshold DDM

1. $T_2 = E[DT^2]$ is the solution of

$$a \frac{dT_2}{dx_0} + \frac{\sigma^2}{2} \frac{d^2 T_2}{dx_0^2} = -2E[DT],$$

with boundary conditions $T_2(\pm z) = 0$.
Double Threshold DDM

1. $T_2 = \mathbb{E}[DT^2]$ is the solution of

$$a \frac{d T_2}{dx_0} + \frac{\sigma^2}{2} \frac{d^2 T_2}{dx_0^2} = -2\mathbb{E}[DT],$$

with boundary conditions $T_2(\pm z) = 0$

2. The variance of decision time is

$$\text{Var} = \frac{\sigma^4}{a^4} \left[3k_z^2 \text{csch}^2(2k_z) - 2k_z^2 e^{-2k_x} \text{csch}(2k_z) \coth(2k_z) - 4k_z k_x e^{-2k_x} \text{csch}(2k_z) \\
- k_z^2 e^{-4k_x} \text{csch}^2(2k_z) + k_z \coth(2k_z) - k_x e^{-2k_x} \text{csch}(2k_z) - k_x \right].$$
Double Threshold DDM: Coefficient of Variation

Coefficient of Variation \( CV = \frac{\sqrt{\text{Var}[DT]}}{E[DT]} \)
Monotonicity of Coefficient of Variation

1. For $k_x = 0$, CV decreases monotonically with $k_z$

2. Double threshold CV converge to single threshold CV for large $k_z$

3. As $k_z \to 0^+$, double threshold CV $\to \sqrt{2/3}$
Computation of Higher Moments: Moment Generating Function Approach

Moment Generating Function

\[ M_X(\alpha) := \mathbb{E}[e^{\alpha X}] = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \mathbb{E}[X^n] \]

Cumulant Generating Function

\[ K_X(\alpha) = \log M_X(\alpha) = \sum_{n=1}^{\infty} \frac{\kappa_n \alpha^n}{n!}, \]

where \( \kappa_1 = \mathbb{E}[X], \ \kappa_2 = \text{Var}[X], \ \kappa_3 = \mathbb{E}[(X - \mathbb{E}[X])^3] \)

For decision time for DDM conditioned on correct decision

\[ K_+(\alpha) = C(a, \sigma, z, x_0) + \log \sinh \left( \frac{(z + x_0) \sqrt{a^2 - 2\alpha \sigma^2}}{\sigma^2} \right) - \log \sinh \left( \frac{2z \sqrt{a^2 - 2\alpha \sigma^2}}{\sigma^2} \right). \]
Conditional Moments of Decision Time

1. Expected DT conditioned on correct response

\[ \mathbb{E}[DT]_+ = \frac{\sigma^2}{a^2} \left( 2k_z \coth(2k_z) - (k_x + k_z) \coth(k_x + k_z) \right). \]

2. Variance of DT conditioned on correct response

\[ \text{Var}_+ = \frac{\sigma^4}{a^4} \left[ 4k_z^2 \text{csch}^2(2k_z) + 2k_z \coth(2k_z) - (k_x + k_z)^2 \text{csch}^2(k_x + k_z) - (k_x + k_z) \coth(k_x + k_z) \right] \]

3. Skewness of DT conditioned on correct response

\[ \text{Skew}_+ \text{Var}_+^{\frac{3}{2}} = \frac{\sigma^6}{a^6} \left[ 12k_z^2 \text{csch}^2(2k_z) + 16k_z^3 \coth(2k_z) \text{csch}^2(2k_z) + 6k_z \coth(2k_z) - 3(k_x + k_z)^2 \text{csch}^2(k_x + k_z) - 2(k_z + k_x)^3 \coth(k_z + k_x) \text{csch}^2(k_z + k_x) - 3(k_z + k_x)^3 \coth(k_z + k_x) \coth(k_z + k_x) \right]. \]

For expressions for DT conditioned on error replace \( k_x \) by \( -k_x \)
1. As $k_z \to 0^+$, $\text{CV}^+ \to \sqrt{2/5}$

2. depending on how $k_x \to 0^−$ as $k_z \to 0^+$, the limit can vary between $\sqrt{2/5}$ and $\sqrt{2/3}$
Unconditional Skewness of Decision Time for DDM

\[
\text{Skew} \times \text{Var}^{3/2} = \frac{\sigma^6}{a^6} \left[ \left( (24k_x k_z^2 + 6k_z^2 - 12k_z^3) e^{-2k_z - 4k_x} + (24k_x^2 k_z + 24k_x k_z - 16k_z^3 + 6k_z) e^{-2k_x} \right) \right.
\]

\[
- (12k_x^2 k_z + 12k_x k_z^2 + 12k_x k_z + 4k_z^3 + 6k_z^2 + 3k_z) e^{4k_z - 2k_x} - (24k_x k_z^2 + 6k_z^2 + 12k_z^3) e^{2k_z - 4k_x}
\]

\[
- 8k_z^3 e^{-6k_x} - 3k_z \cosh(2k_z) + 3k_z \cosh(6k_z) + 9k_x \sinh(2k_z) - 3k_x \sinh(6k_z) + 56k_z^3 \cosh(2k_z)
\]

\[
+ 36k_z^2 \sinh(2k_z) - (3k_z - 6k_z^2 + 4k_z^3 + 12k_x k_z - 12k_x k_z^2 + 12k_x^2 k_z) e^{-4k_z - 2k_x} \right) \frac{\cosh(2k_z)}{4} \right].
\]
O-U as a transformation of the Wiener Process

1. Consider the O-U process:

\[ dX(t) = adt - \theta Xdt + \sigma dW(t). \]

2. \( X(t) \) can be written as

\[ X(t) = \frac{a}{\theta} (1 - e^{-\theta t}) + e^{-\theta t} x_0 + e^{-\theta t} W(\sigma^2(e^{2\theta t} - 1) / 2\theta). \]
O-U as a transformation of the Wiener Process

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3. If \( \tau \) is decision time for O-U process with respect to boundary \( z \), then

   \[ s = \sigma^2(e^{2\theta \tau} - 1)/2\theta \]

   is the decision time for Wiener process \( W(u) \) starting as \( x_0 - \frac{a}{\theta} \) with respect to boundary

   \[ \tilde{z}_1(u) := \frac{\sqrt{\sigma^2 + 2\theta u}}{\sigma}(z - \frac{a}{\theta}) \]
O-U as a transformation of the Wiener Process